Exam 1 Review

MA 265

February 2021

Below is a short collection of some of the results mentioned so far in the course. These have been condensed as much as possible and some results have been combined in larger equivalence statements.

Theorem 1. *Suppose the matrix equation* $A = b$ *is consistent and let* p *be any solution. The complete solution set of* $Ax = b$ *consists of vectors of the form* $w = p + v$ *where* v *is a solution to the equation* $Ax = 0$.

Theorem 2. *A set of vectors* $\{v_1, \dots, v_\ell\}$ *for* $\ell \geq 2$ *is linearly dependent if and only if at least one these vectors is a linear combination of the others.*

Theorem 3. *Suppose* $\{v_1, \dots, v_\ell\}$ *is a collection of vectors of* $\mathbb{R}^n$. *If* $\ell > n$, *then this set of vectors is linearly dependent.*

Theorem 4. *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. There exists an* $m \times n$ *matrix* A *such that* $T(x) = Ax$ *for all* x *in* $\mathbb{R}^n$. *Moreover,* $M = [T(e_1)|\cdots|T(e_n)]$ *where* $e_1, \dots, e_n$ *are the columns of* I_n.

Theorem 5. *Suppose* $T_1 : \mathbb{R}^p \to \mathbb{R}^n$ *is a linear transformation whose matrix is* B *(*$T(x) = Bx$*) and suppose* $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation whose matrix is* A *(*$T(y) = Ay$*). Then* $T_2 \circ T_1 : \mathbb{R}^p \to \mathbb{R}^n$ *is a linear transformation is whose matrix is* AB.

Theorem 6. *Let* A *be an* $m \times n$ *matrix and* $T : \mathbb{R}^n \to \mathbb{R}^m$ *defined by* A, $T(x) = Ax$. *The following are equivalent:*

(a) T *is one-to-one*

(b) The columns of A *are linearly independent*

(c) $Ax = 0$ *has only the trivial solution (equivalently* $T(x) = 0$ *has only the trivial solution).*

(d) $\mathrm{Nul}(A) = \{0\}$

Theorem 7. *Let* A *be an* $m \times n$ *matrix and* $T : \mathbb{R}^n \to \mathbb{R}^m$ *defined by* A, $T(x) = Ax$. *The following are equivalent:*

(a) T *is onto*

(b) Every b *of* $\mathbb{R}^m$ *can be written as a linear combination of the columns of* A.

(c) $Ax = b$ *has a solution for all* b *of* $\mathbb{R}^m$.

(d) A *has a pivot position in every row.*

(e) $\mathrm{Col}(A) = \mathbb{R}^m$

Theorem 8. *Suppose*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 *matrix. This matrix* A *is invertible if and only if* $ad - bc \neq 0$ *and in this case*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 9. *Suppose* A *is an invertible* $n \times n$ *matrix. For all* b *in* $\mathbb{R}^n$*, the matrix equation* $Ax = b$ *has a unique solution and moreover, this solution is* $x = A^{-1}b$.

Theorem 10. *Let* A *be an* $n \times n$ ***square*** *matrix and* $T : \mathbb{R}^n \to \mathbb{R}^n$ *defined by* A, $T(x) = Ax$. *The following are equivalent:*

(a) A *is invertible.*

(b) A *is row equivalent to* I_n.

(c) A *has* n *pivot positions*

(d) $Ax = 0$ *has only the trivial solution*

(e) The columns of A *are linearly independent*

(f) T *is one-to-one*

(g) $Ax = b$ *has a unique solutions for all* b *in* $\mathbb{R}^n$

(h) the columns of A *span* $\mathbb{R}^n$

(i) T *is onto*

(j) A^T *is invertible*

Properties of transposes and Inverses

(a) If A and B are both $m \times n$ matrices $(A + B)^T = A^T + B^T$.

(b) For any real number r, $(r \cdot A)^T = r \cdot A^T$.

(c) $(A^T)^T = A$

(d) If A is an $m \times n$ matrix and B is a $n \times p$ matrix, then $(AB)^T = B^T A^T$.

(e) Suppose A and B are $n \times n$ invertible matrices:

(i) $(A^{-1})^{-1} = A$

(ii) If r is any nonzero scalar, then $(r \cdot A)^{-1} = \frac{1}{r} \cdot A^{-1}$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^T)^{-1} = (A^{-1})^T$

Theorem 11. *If A is any matrix, the pivot columns of A form a basis for the column space of A,* $\mathrm{Col}\,(A)$*.*

Theorem 12 (Rank Theorem)**.** *Let A be an $m \times n$ matrix. Then* $\mathrm{rank}\, A + \dim \mathrm{Nul}(A) = n$

1. Consider the matrix A= $\begin{bmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{bmatrix}$

(a) Row-reduce A to echelon form.

(b) Row-reduce A to reduced echelon form.

(c) Identify the pivot positions and pivot columns of A.

(d) If $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is $Ax = b$ consistent? If so, describe the solution set in parametric form.

(e) Are the columns of A linearly independent?

(f) Do the columns of A span $\mathbb{R}^3$?

(g) if T is the linear transformation of A ($T(x) = Ax$), is T one-to-one? Is T onto?

(h) Find a basis for $\mathrm{Nul}(A)$

(i) Find a basis for $\mathrm{Col}(A)$.

(j) What is $\dim \mathrm{Nul}(A)$?

(k) what is $\mathrm{rank}\, A$?

2. Repeat (a)-(k) above with $A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$.

3. Repeat (a)-(k) above with $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & -3 \end{bmatrix}$.

4. What is the largest number of pivots a 4×6 matrix can have?

5. What is the largest number of pivots a 6×4 matrix can have?

6. Is $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ contained in Span $\left\{ \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$?

7. Are the following vectors linearly independent or linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

If linearly dependent, provide an equation of linear dependence.

8. Let $\{v_1, \ldots, v_n\}$ be a set of vectors of $\mathbb{R}^m$.

(a) Can these vectors be linearly independent if $m > n$?

(b) Can these vectors be linearly independent if $m < n$?

(c) Can these vectors span $\mathbb{R}^m$ if $m > n$?

(d) Can these vectors span $\mathbb{R}^m$ if $m < n$?

If "yes" to any of the above is this always the case? As in, just because it is possible, must it always occur?

9. Compute the (2,3)-entry of the inverse of $\begin{bmatrix} 3 & 4 & 2 \\ 5 & 0 & -6 \\ -1 & 2 & 0 \end{bmatrix}$ without computing the entire inverse.

10. Now compute the inverse of $\begin{bmatrix} 3 & 4 & 2 \\ 5 & 0 & -6 \\ -1 & 2 & 0 \end{bmatrix}$

11. Calculate the area of the area of the parallelogram with vertices (-1,1), (-2,-2), (2,-1) and (3,2).

12. suppose the reduced echelon form for a matrix A is

$$A = \begin{bmatrix} 1 & 0 & 0 & -3 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What are rank(A) and dim Nul (A)?

13. Is $B = \begin{bmatrix} 2 & 7 \\ 5 & 1 \end{bmatrix}$ invertible? If yes, what is its inverse?

14. Let $A = \begin{bmatrix} 2 & 6 & 7 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$. What's $\det(A^5)$?

15. Let T be the linear transformation with standard matrix $\begin{bmatrix} 1 & -3 & 6 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$. Which of the following statements are true?

(a) T maps $\mathbb{R}^3$ to $\mathbb{R}^4$

(b) T maps $\mathbb{R}^4$ to $\mathbb{R}^3$

(c) T is onto

(d) T is one-to-one

16. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-3\\6\\13\\23\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\12\\-1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\6\\-14\\8\end{bmatrix}$$

What's the matrix of T?

17. Compute the determinant of

$$\begin{bmatrix}4 & 6 & 3 & 2\\0 & 2 & 8 & 19\\0 & 0 & 1 & -13\\0 & 0 & 0 & -2\end{bmatrix} \text{ and of } \begin{bmatrix}1 & 5 & 9 & 13\\2 & 6 & 10 & 14\\3 & 7 & 11 & 15\\4 & 8 & 12 & 16\end{bmatrix}$$

18. Find a basis for the column space and a basis for the null space of

$$A = \begin{bmatrix}1 & 0 & -2\\3 & 1 & -2\\5 & 1 & -6\end{bmatrix}$$

Is A invertible?

19. Find the matrices for the following linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, the linear transformation that rotates points in $\mathbb{R}^2$ counterclockwise about the origin by $\frac{\pi}{6}$ radians.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, the linear transformation that leaves e_1 unchanged and maps e_2 to e_1 $-2\ e_2$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$, the linear transformation that rotates points in $\mathbb{R}^2$ counterclockwise about the origin by $\frac{\pi}{4}$ radians and then flips over the vertical x_2-axis.